

OSNOVI METODE KONAČNIH ELEMENATA Predavanje VII



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OSNOVI METODE KONAČNIH ELEMENATA

Predavanje VII



3D problemi

Prirodne koordinate i interpolacione funkcije

KE oblika tetraedra

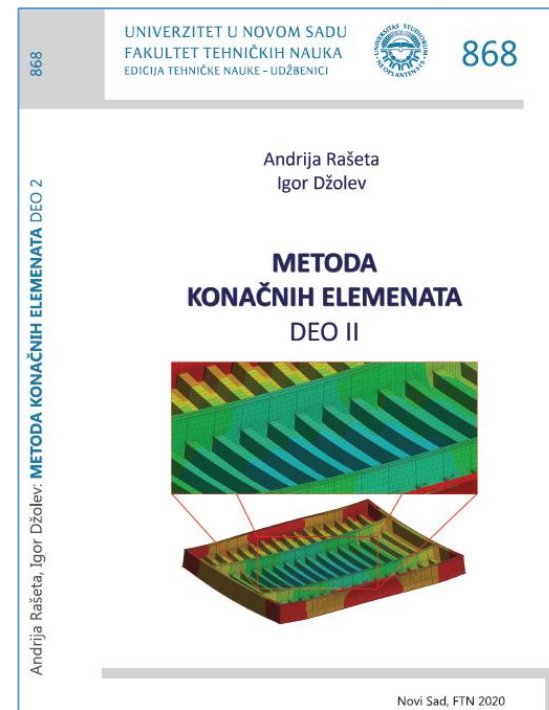
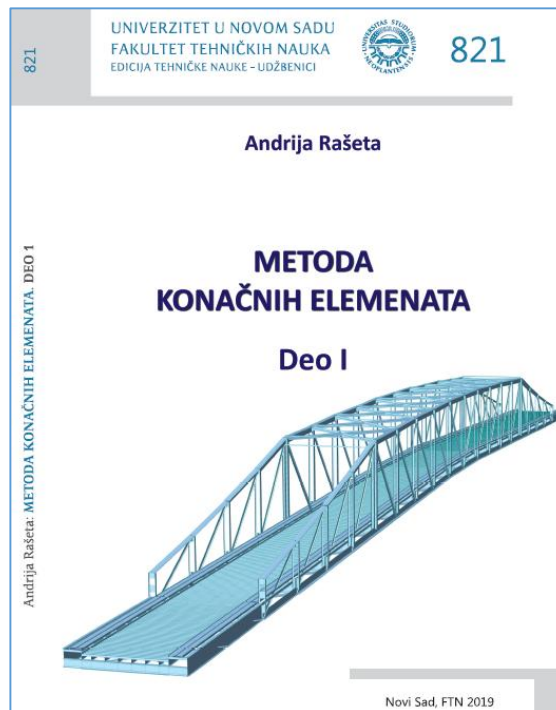
KE oblika heksaedra

Izoparametarski 3D KE

Linearna statička analiza

Literatura

- **Metoda konačnih elemenata, deo I,**
A. Rašeta, FTN Novi Sad, 2019.
- **Metoda konačnih elemenata, deo II,**
A. Rašeta, I. Džolev, FTN Novi Sad, 2020.



Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika tetraedra**
- Čvorovi tetraedra u Dekartovom koordinatnom sistemu desne orijentacije obeleženi su tako da rastu suprotno od smera obrtanja kazaljke na časovniku gledajući sa strane čvora 4
- Položaj proizvoljne tačke u zavisnosti od koordinata temena može da se prikaže na sledeći način

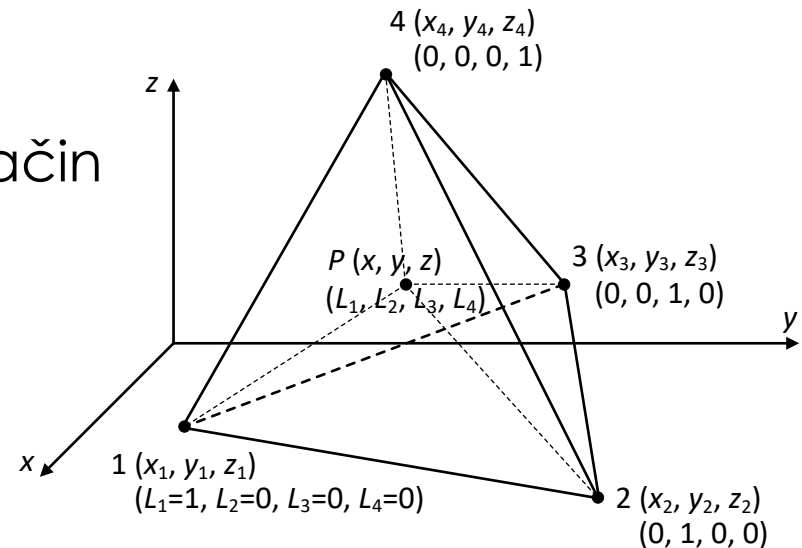
$$x = L_1 x_1 + L_2 x_2 + L_3 x_3 + L_4 x_4$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3 + L_4 y_4$$

$$z = L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4$$

- pri čemu **zapreminske koordinate** nisu međusobno nezavisne

$$L_1 + L_2 + L_3 + L_4 = 1$$



Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika tetraedra

■ Prethodni izrazi u matričnom obliku glase

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

■ odakle sledi

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \quad V = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \frac{1}{6}(a_1 + a_2 + a_3 + a_4)$$

$$L_i = \frac{1}{6V} (a_i + b_i x + c_i y + d_i z), \quad i = 1, 2, 3, 4$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \quad b_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, \quad c_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, \quad d_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

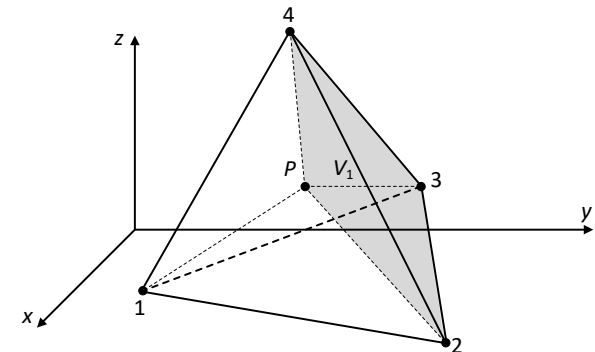
■ KE oblika tetraedra

- Izraz u zagradi uz koeficijent $1/6$ za $i = 1$ predstavlja zapreminu dela tetraedra definisanog koordinatama čvorova $2(x_2, y_2)$, $3(x_3, y_3)$, $4(x_4, y_4)$ i tačke $P(x, y, z)$ pa se na osnovu toga zaključuje da zapreminska koordinata L_1 predstavlja odnos zapremine dela tetraedra definisanog čvorovima $P, 2, 3, 4$ naspram čvora 1 i zapremine V celog tetraedra

$$L_1 = \frac{V_1}{V} \quad L_2 = \frac{V_2}{V} \quad L_3 = \frac{V_3}{V} \quad L_4 = \frac{V_4}{V}$$

- S obzirom na to da se vrednost zapreminske koordinate L_i linearno menja od 1 u i -tom čvoru (temenu) do 0 u preostalim čvorovima (temenima) ona je jednaka IF i -tog čvora osnovnog tetraedarskog KE (linearna interpolacija), odnosno važi sledeće

$$N_1 = L_1 \quad N_2 = L_2 \quad N_3 = L_3 \quad N_4 = L_4$$



Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika tetraedra

- Ako je f funkcija od L_1, L_2, L_3 i L_4 operacija parcijalnog diferenciranja po koordinatama x, y i z obavlja se na sledeći način

$$\frac{\partial f}{\partial x} = \sum_{i=1}^4 \frac{\partial f}{\partial L_i} \frac{\partial L_i}{\partial x}, \quad \frac{\partial f}{\partial y} = \sum_{i=1}^4 \frac{\partial f}{\partial L_i} \frac{\partial L_i}{\partial y}, \quad \frac{\partial f}{\partial z} = \sum_{i=1}^4 \frac{\partial f}{\partial L_i} \frac{\partial L_i}{\partial z}$$

$$\frac{\partial L_i}{\partial x} = \frac{b_i}{6V}, \quad \frac{\partial L_i}{\partial y} = \frac{c_i}{6V}, \quad \frac{\partial L_i}{\partial z} = \frac{d_i}{6V}, \quad i = 1, 2, 3, 4$$

- Integracija po zapremini KE kada se pod integralom javljaju prirodne koordinate obavlja se pomoću izraza

$$\int_V L_1^i(x, y, z) L_2^j(x, y, z) L_3^k(x, y, z) L_4^l(x, y, z) dV = \frac{i! j! k! l!}{(i + j + k + l + 3)!} 6V$$

- gde su i, j, k i l celobrojni eksponenti

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

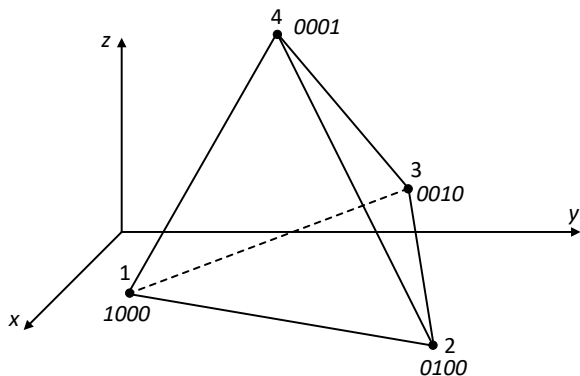
- KE oblika tetraedra
- IF mogu da se odrede pomoću formule

$$N_{\alpha\beta\gamma\delta}(L_1, L_2, L_3, L_4) = N_{\alpha}(L_1)N_{\beta}(L_2)N_{\gamma}(L_3)N_{\delta}(L_4)$$

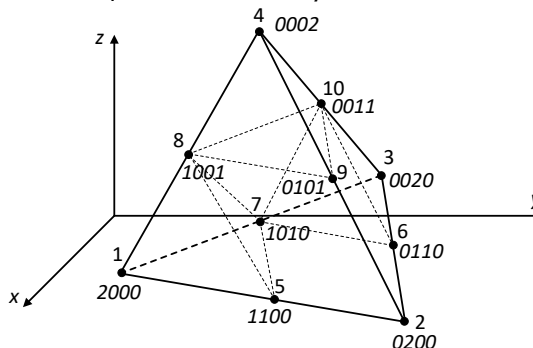
$$N_{\alpha}(L_1) = \begin{cases} \prod_{i=1}^{\alpha} \left(\frac{nL_1 - i + 1}{i} \right) & \text{za } \alpha \geq 1 \\ 1 & \text{za } \alpha = 0 \end{cases}$$

- pri čemu je n stepen polinoma
- Analogni izrazi važe za $N_{\beta}(L_2)$, $N_{\gamma}(L_3)$ i $N_{\delta}(L_4)$
- Čvorovima KE pridružuju se četiri broja čiji je zbir jednak stepenu interpolacionog polinoma

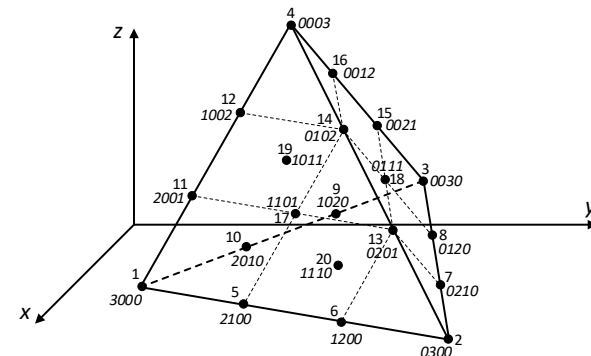
Linearna interpolacija
(osnovni element; 4 čvora;
12 stepeni slobode)



Kvadratna interpolacija (element drugog reda; 10 čvorova (u temenima i sredinama izvodnica), 30 stepeni slobode)



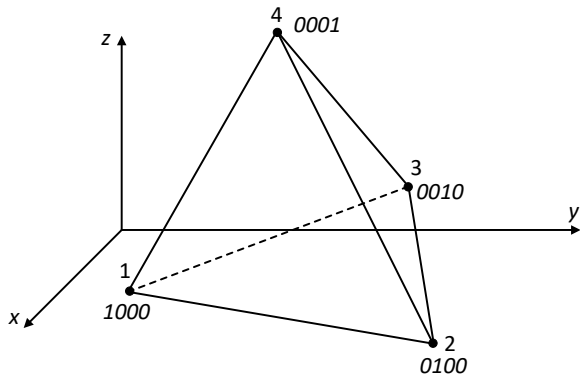
Kubna interpolacija (element trećeg reda; 20 čvorova (u temenima, trećinama izvodnica i težištima površina strana), 60 stepeni slobode)



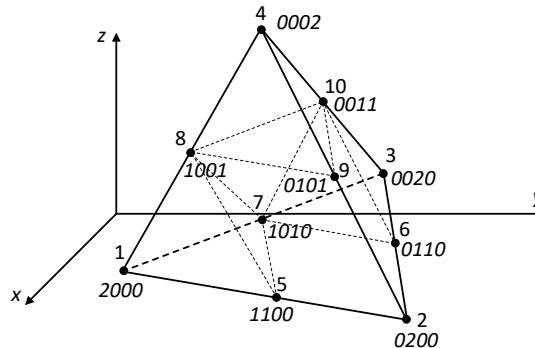
Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- KE oblika tetraedra
- IF mogu da se odrede pomoću formule
 - Svaki broj se odnosi na jednu od stranica tetraedra. Prvi broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_1 = 0$ (naspram čvora 1), drugi broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_2 = 0$ (naspram čvora 2), treći broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_3 = 0$ (naspram čvora 3) i četvrti broj je jednak 0 u čvorovima koji leže na površini strane gde je $L_4 = 0$ (naspram čvora 4). Svaki od brojeva koji leži na paralelnim ravnima (koje su na jednakim međusobnim rastojanjima) u odnosu na odgovarajuću stranu tetraedra rastu sa udaljavanjem od ravni na kojoj je odgovarajuća zapreminska koordinata jednaka nuli

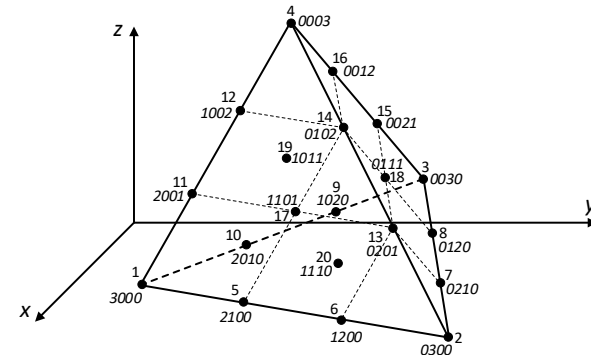
Linearna interpolacija
(osnovni element; 4 čvora;
12 stepeni slobode)



Kvadratna interpolacija (element drugog reda; 10 čvorova (u temenima i sredinama izvodnica), 30 stepeni slobode)



Kubna interpolacija (element trećeg reda; 20 čvorova (u temenima, trećinama izvodnica i težištima površina strana), 60 stepeni slobode)



Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika tetraedra

■ Linearne IF

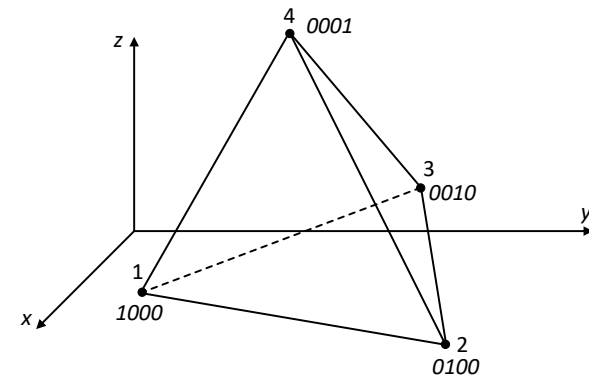
■ čvor 1000 $N_\alpha(L_1) = N_{\alpha=1}(L_1) = \left(\frac{1 \cdot L_1 - 1 + 1}{1} \right)$

$$N_\beta(L_2) = N_{\beta=0}(L_2) = 1$$

$$N_\gamma(L_3) = N_{\gamma=0}(L_3) = 1$$

$$N_\delta(L_4) = N_{\delta=0}(L_4) = 1$$

$$N_1 = L_1$$



- Analognim postupkom određuju se ostale IF

$$N_2 = L_2 \quad N_3 = L_3 \quad N_4 = L_4$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

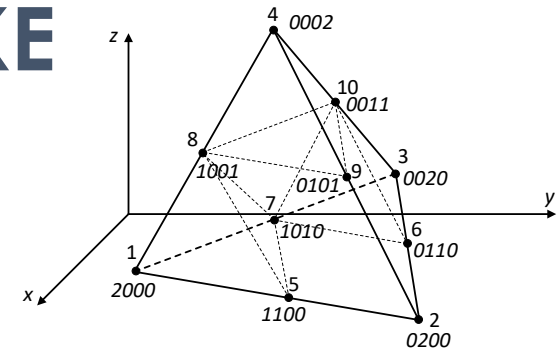
- KE oblika tetraedra
- Kvadratne IF

- čvor 2000 $N_{\alpha}(L_1) = N_{\alpha=2}(L_1) = \left(\frac{2 \cdot L_1 - 1 + 1}{1}\right) \left(\frac{2 \cdot L_1 - 2 + 1}{2}\right)$
 $N_{\beta}(L_2) = N_{\beta=0}(L_2) = 1$
 $N_{\gamma}(L_3) = N_{\gamma=0}(L_3) = 1$
 $N_{\delta}(L_4) = N_{\delta=0}(L_4) = 1$

- čvor 1100 $N_{\alpha}(L_1) = N_{\alpha=1}(L_1) = \left(\frac{2 \cdot L_1 - 1 + 1}{1}\right)$
 $N_{\beta}(L_2) = N_{\beta=1}(L_2) = \left(\frac{2 \cdot L_2 - 1 + 1}{1}\right)$
 $N_{\gamma}(L_3) = N_{\gamma=0}(L_3) = 1$
 $N_{\delta}(L_4) = N_{\delta=0}(L_4) = 1$

- Analognim postupkom određuju se ostale IF

$$\begin{array}{llll}
 N_2 = L_2(2L_2 - 1) & N_3 = L_3(2L_3 - 1) & N_4 = L_4(2L_4 - 1) & N_6 = 4L_24L_3 \\
 N_7 = 4L_24L_3 & N_8 = 4L_14L_3 & N_9 = 4L_24L_4 & N_{10} = 4L_34L_4
 \end{array}$$



$$N_1 = L_1(2L_1 - 1)$$

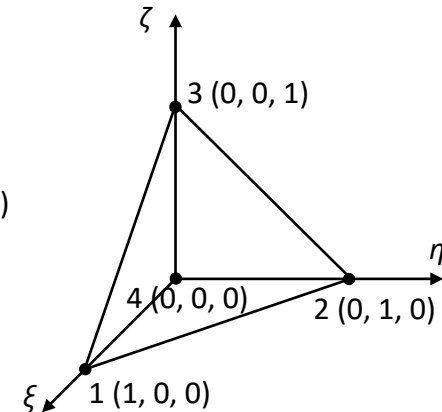
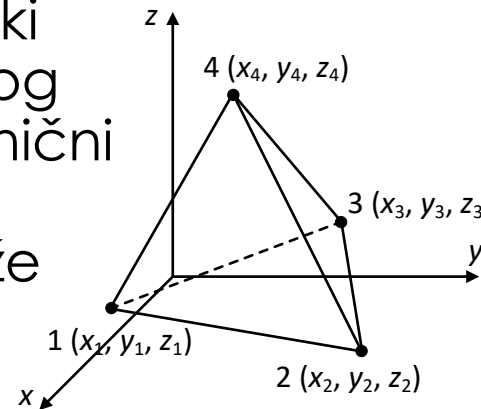
$$N_5 = 4L_14L_2$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika tetraedra

- Analogno kao i kod dvodimenzionalnih elemenata pored prethodno opisanih prirodnih zapreminskih koordinata može da se koristi **sistem prirodnih koordinata ξ , η i ζ**

- Proizvoljni osnovni tetraedarski KE iz Dekartovog koordinatnog sistema preslikava se na jedinični tetraedar u prirodnom koordinatnom sistemu, tj. važe sledeće veze za linearnu transformaciju



$$x = x_4 + (x_1 - x_4)\xi + (x_2 - x_4)\eta + (x_3 - x_4)\zeta$$

$$y = y_4 + (y_1 - y_4)\xi + (y_2 - y_4)\eta + (y_3 - y_4)\zeta$$

$$z = z_4 + (z_1 - z_4)\xi + (z_2 - z_4)\eta + (z_3 - z_4)\zeta$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika tetraedra

- Ako je f funkcija od x, y i z koje su u funkciji od ξ, η i ζ , tj. $f = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta))$ operacija parcijalnog diferenciranja obavlja se na sledeći način

$$\begin{aligned} \frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \zeta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \zeta} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \zeta} \end{aligned} \quad \left\{ \begin{array}{c} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} \end{array} \right\} = \left[\begin{array}{ccc} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{array} \right\} = \mathbf{J} \left\{ \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{array} \right\}$$

$$\mathbf{J} = \left[\begin{array}{ccc} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{array} \right] = \left[\begin{array}{ccc} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{array} \right] = \left[\begin{array}{ccc} x_1 - x_4 & x_2 - x_4 & x_3 - x_4 \\ y_1 - y_4 & y_2 - y_4 & y_3 - y_4 \\ z_1 - z_4 & z_2 - z_4 & z_3 - z_4 \end{array} \right]$$

$$\left\{ \begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{array} \right\} = \mathbf{J}^{-1} \left\{ \begin{array}{c} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \\ \frac{\partial f}{\partial \zeta} \end{array} \right\} \quad \mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \left[\begin{array}{ccc} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{array} \right]$$

$$\det \mathbf{J} = -J_{13}J_{22}J_{31} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{11}J_{23}J_{32} - J_{12}J_{21}J_{33} + J_{11}J_{22}J_{33}$$

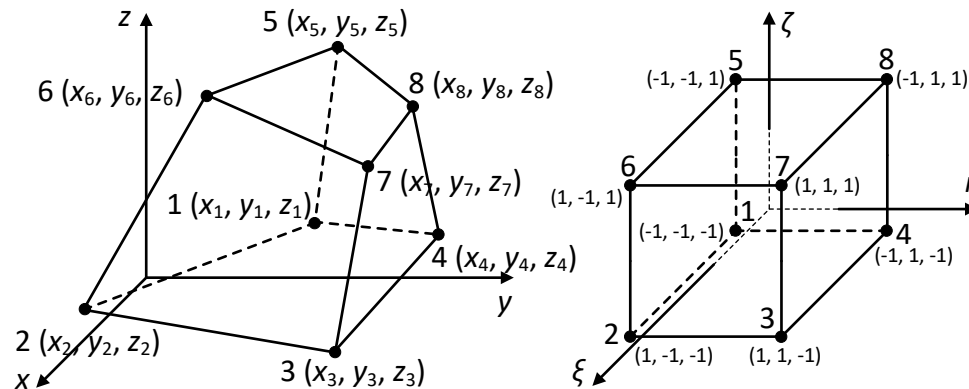
$$v = \frac{1}{6} \det \mathbf{J}$$

$$L_1 = \xi, \quad L_2 = \eta, \quad L_3 = \zeta, \quad L_4 = 1 - \xi - \eta - \zeta$$

$$dx dy dz = \det \mathbf{J} d\xi d\eta d\zeta$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

- **KE oblika heksaedra**
- Uvodi se sistem **prirodnih koordinata ξ, η i ζ** koji je analogan sistemu prirodnih koordinata za element u obliku četvorougla



- Dekartove koordinate x, y i z proizvoljne tačke u polju KE mogu da se prikažu kao linearna kombinacija koordinata čvorova $x_1, y_1, \dots, x_8$ i y_8

$$x = \sum_{i=1}^8 L_i x_i \quad y = \sum_{i=1}^8 L_i y_i \quad z = \sum_{i=1}^8 L_i z_i$$

Prirodne koordinate i interpolacione funkcije. Trodimenzionalni KE

■ KE oblika heksaedra

- Za određivanje funkcija L_i koristi se veza između globalnih Dekartovih i prirodnih (lokalnih bezdimenzionalnih) koordinata

$$x = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \zeta + \alpha_5 \xi \eta + \alpha_6 \xi \zeta + \alpha_7 \eta \zeta + \alpha_8 \xi \eta \zeta$$

$$x = \mathbf{A} \boldsymbol{\alpha} \quad \mathbf{A} = [1 \quad \xi \quad \eta \quad \zeta \quad \xi \eta \quad \xi \zeta \quad \eta \zeta \quad \xi \eta \zeta]$$

$$\boldsymbol{\alpha}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8\}$$

- Za temena heksaedra važi

$$x = x_i, \quad \xi = \xi_i, \quad \eta = \eta_i, \quad i = 1, 2, 3, \dots, 8$$

- Koristeći prethodne za svaki čvor KE sledi

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix}$$

$$\mathbf{X} = \mathbf{E} \boldsymbol{\alpha} \quad \boldsymbol{\alpha} = \mathbf{E}^{-1} \mathbf{X} \quad x = \mathbf{A} \mathbf{E}^{-1} \mathbf{X} \quad \mathbf{L} = \mathbf{A} \mathbf{E}^{-1}$$

$$L_i = \frac{1}{8} (1 + \xi_i \xi) (1 + \eta_i \eta) (1 + \zeta_i \zeta), \quad i = 1, 2, 3, \dots, 8$$

$$\mathbf{E}^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Analizira se KE oblika tetraedra sa čvorovima u temenima koji su obeleženi tako da njihovi brojevi 1, 2 i 3 rastu u smeru suprotnom od smera obrtanja kazaljke na časovniku posmatrajući sa strane čvora 4

$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4\}$$

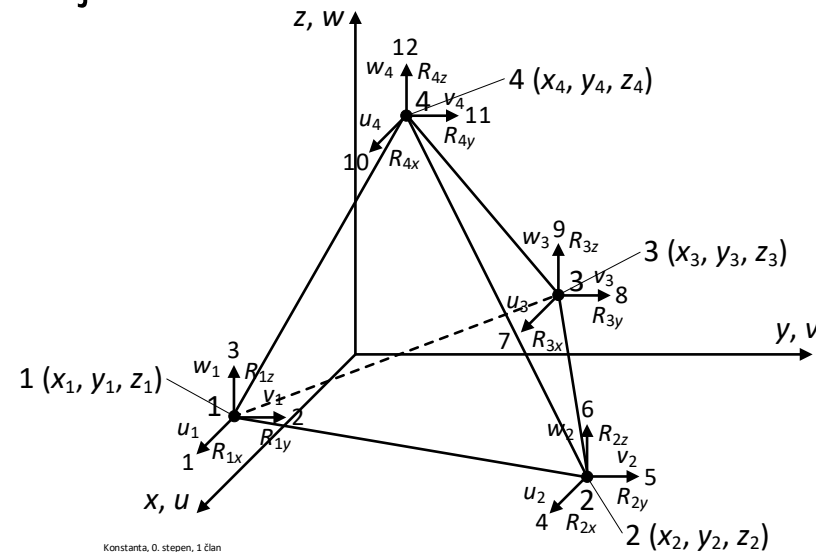
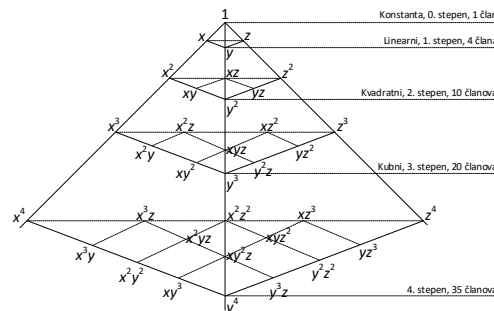
$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \quad i = 1, 2, 3, 4$$

- Raspodela pomeranja u polju KE definisana je potpunim polinomima prvog stepena

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 z$$

$$w = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z$$



KE oblika tetraedra. Linearna interpolacija. CSTh KE

■ IF. Direktan postupak

$$\boldsymbol{\alpha}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8 \quad \alpha_9 \quad \alpha_{10} \quad \alpha_{11} \quad \alpha_{12}\}$$

$$\mathbf{u} = \mathbf{A}\boldsymbol{\alpha} \rightarrow \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & z \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{12} \end{Bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & x & y & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & z \end{bmatrix}$$

- Za čvorove KE (granični uslovi su: $u = u_i$, $v = v_i$, $w = w_i$ i koordinate čvorova su: $x = x_i$, $y = y_i$, $z = z_i$, gde je $i = 1, 2, 3, \text{ i } 4$) sledi

$$\mathbf{d} = \mathbf{C}\boldsymbol{\alpha} \rightarrow \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \end{Bmatrix} \quad \text{itd.}$$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

■ IF. Zapreminske koordinate

$$\mathbf{N} = \mathbf{A}\mathbf{C}^{-1} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}$$

$$N_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z), \quad i = 1, 2, 3, 4$$

$$V = \frac{1}{6} \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = \frac{1}{6}(a_1 + a_2 + a_3 + a_4)$$

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}, \quad b_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}, \quad c_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix}, \quad d_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

$$a_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \quad b_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}, \quad c_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix}, \quad d_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

Matrica **B**

$$\mathbf{B} = \mathbf{D}_k \mathbf{N} \quad \mathbf{D}_k = \begin{bmatrix} \partial/\partial_x & 0 & 0 \\ 0 & \partial/\partial_y & 0 \\ 0 & 0 & \partial/\partial_z \\ \partial/\partial_y & \partial/\partial_x & 0 \\ 0 & \partial/\partial_z & \partial/\partial_y \\ \partial/\partial_z & 0 & \partial/\partial_x \end{bmatrix} \quad \mathbf{B} = \frac{1}{6V} \begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 \\ 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 & 0 \\ 0 & 0 & d_1 & 0 & 0 & d_2 & 0 & 0 & d_3 & 0 & 0 & d_4 \\ c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 & 0 \\ 0 & d_1 & c_1 & 0 & d_2 & c_2 & 0 & d_3 & c_3 & 0 & d_4 & c_4 \\ d_1 & 0 & b_1 & d_2 & 0 & b_2 & d_3 & 0 & b_3 & d_4 & 0 & b_4 \end{bmatrix}$$

Matrica krutosti

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV = \mathbf{B}^T \mathbf{D} \mathbf{B} V \quad \mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}$$

Komentari:

- Linearna raspodela komponenta pomeranja po spoljašnjim površinama tetraedra može jednoznačno da se odredi sa tri stepena slobode pomeranja u čvorovima pa se zaključuje da je kompatibilnost pomeranja na površinama između susednih KE zadovoljena, tj. element **spada u konformne KE**
- Takođe, KE **ispunjava i ostale uslove za monotonu konvergenciju** rešenja (interpolacione funkcije mogu da opišu pomeranje kao krutog tela i polje konstantnih deformacija)
- S obzirom na to da su elementi matrice **B** konstantni sledi da su komponente deformacije u KE konstantne pa se element naziva **tetraedar sa konstantnim deformacijama ili CSTh element** (Constant Strain Tetrahedral element)

KE oblika tetraedra. Linearna interpolacija. CSTh KE

■ Matrica krutosti

$$k = \frac{1}{36V}$$

kolone 1 do 3

$b_1^2 D_{11} + c_1^2 D_{44} + d_1^2 D_{66}$	$b_1 c_1 (D_{12} + D_{44})$	$b_1 d_1 (D_{13} + D_{66})$
$b_1 c_1 (D_{21} + D_{44})$	$c_1^2 D_{22} + b_1^2 D_{44} + d_1^2 D_{55}$	$c_1 d_1 (D_{23} + D_{55})$
$b_1 d_1 (D_{31} + D_{66})$	$c_1 d_1 (D_{32} + D_{55})$	$d_1^2 D_{33} + c_1^2 D_{55} + b_1^2 D_{66}$
$b_1 b_2 D_{11} + c_1 c_2 D_{44} + d_1 d_2 D_{66}$	$b_2 c_1 D_{12} + b_1 c_2 D_{44}$	$b_2 d_1 D_{13} + b_1 d_2 D_{66}$
$b_1 c_2 D_{21} + b_2 c_1 D_{44}$	$c_1 c_2 D_{22} + b_1 b_2 D_{44} + d_1 d_2 D_{55}$	$c_2 d_1 D_{23} + c_1 d_2 D_{55}$
$b_1 d_2 D_{31} + b_2 d_1 D_{66}$	$c_1 d_2 D_{32} + c_2 d_1 D_{55}$	$d_1 d_2 D_{33} + c_1 c_2 D_{55} + b_1 b_2 D_{66}$
$b_1 b_3 D_{11} + c_1 c_3 D_{44} + d_1 d_3 D_{66}$	$b_3 c_1 D_{12} + b_1 c_3 D_{44}$	$b_3 d_1 D_{13} + b_1 d_3 D_{66}$
$b_1 c_3 D_{21} + b_3 c_1 D_{44}$	$c_1 c_3 D_{22} + b_1 b_3 D_{44} + d_1 d_3 D_{55}$	$c_3 d_1 D_{23} + c_1 d_3 D_{55}$
$b_1 d_3 D_{31} + b_3 d_1 D_{66}$	$c_1 d_3 D_{32} + c_3 d_1 D_{55}$	$d_1 d_3 D_{33} + c_1 c_3 D_{55} + b_1 b_3 D_{66}$
$b_1 b_4 D_{11} + c_1 c_4 D_{44} + d_1 d_4 D_{66}$	$b_4 c_1 D_{12} + b_1 c_4 D_{44}$	$b_4 d_1 D_{13} + b_1 d_4 D_{66}$
$b_1 c_4 D_{21} + b_4 c_1 D_{44}$	$c_1 c_4 D_{22} + b_1 b_4 D_{44} + d_1 d_4 D_{55}$	$c_4 d_1 D_{23} + c_1 d_4 D_{55}$
$b_1 d_4 D_{31} + b_4 d_1 D_{66}$	$c_1 d_4 D_{32} + c_4 d_1 D_{55}$	$d_1 d_4 D_{33} + c_1 c_4 D_{55} + b_1 b_4 D_{66}$

kolone 7 do 9

$b_1 b_3 D_{11} + c_1 c_3 D_{44} + d_1 d_3 D_{66}$	$b_1 c_3 D_{12} + b_3 c_1 D_{44}$	$b_1 d_3 D_{13} + b_3 d_1 D_{66}$
$b_3 c_1 D_{21} + b_1 c_3 D_{44}$	$c_1 c_3 D_{22} + b_1 b_3 D_{44} + d_1 d_3 D_{55}$	$c_1 d_3 D_{23} + c_3 d_1 D_{55}$
$b_3 d_1 D_{31} + b_1 d_3 D_{66}$	$c_3 d_1 D_{32} + c_1 d_3 D_{55}$	$d_1 d_3 D_{33} + c_1 c_3 D_{55} + b_1 b_3 D_{66}$
$b_2 b_3 D_{11} + c_2 c_3 D_{44} + d_2 d_3 D_{66}$	$b_2 c_3 D_{12} + b_3 c_2 D_{44}$	$b_2 d_3 D_{13} + b_3 d_2 D_{66}$
$b_3 c_2 D_{21} + b_2 c_3 D_{44}$	$c_2 c_3 D_{22} + b_2 b_3 D_{44} + d_2 d_3 D_{55}$	$c_2 d_3 D_{23} + c_3 d_2 D_{55}$
$b_3 d_2 D_{31} + b_2 d_3 D_{66}$	$c_3 d_2 D_{32} + c_2 d_3 D_{55}$	$d_2 d_3 D_{33} + c_2 c_3 D_{55} + b_2 b_3 D_{66}$
$b^2 D_{11} + c^2 D_{44} + d^2 D_{66}$	$b_3 c_3 (D_{12} + D_{44})$	$b_3 d_3 (D_{13} + D_{66})$
$b_3 c_3 (D_{21} + D_{44})$	$c_3^2 D_{22} + b_3^2 D_{44} + d_3^2 D_{55}$	$c_3 d_3 (D_{23} + D_{55})$
$b_3 d_3 (D_{31} + D_{66})$	$c_3 d_3 (D_{32} + D_{55})$	$d_3^2 D_{33} + c_3^2 D_{55} + b_3^2 D_{66}$
$b_3 b_4 D_{11} + c_3 c_4 D_{44} + d_3 d_4 D_{66}$	$b_4 c_3 D_{12} + b_3 c_4 D_{44}$	$b_4 d_3 D_{13} + b_3 d_4 D_{66}$
$b_3 c_4 D_{21} + b_4 c_3 D_{44}$	$c_3 c_4 D_{22} + b_3 b_4 D_{44} + d_3 d_4 D_{55}$	$c_4 d_3 D_{23} + c_3 d_4 D_{55}$
$b_3 d_4 D_{31} + b_4 d_3 D_{66}$	$c_3 d_4 D_{32} + c_4 d_3 D_{55}$	$d_3 d_4 D_{33} + c_3 c_4 D_{55} + b_3 b_4 D_{66}$

kolone 4 do 6

$b_1 b_2 D_{11} + c_1 c_2 D_{44} + d_1 d_2 D_{66}$	$b_2 c_1 D_{12} + b_1 c_2 D_{44}$	$b_1 d_1 D_{13} + b_2 d_2 D_{66}$
$b_2 c_1 D_{21} + b_1 c_2 D_{44}$	$c_1 c_2 D_{22} + b_1 b_2 D_{44} + d_1 d_2 D_{55}$	$c_1 d_2 D_{23} + c_2 d_1 D_{55}$
$b_2 d_1 D_{31} + b_1 d_2 D_{66}$	$c_2 d_1 D_{32} + c_1 d_2 D_{55}$	$d_1 d_2 D_{33} + c_1 c_2 D_{55} + b_1 b_2 D_{66}$
$b_2^2 D_{11} + c_2^2 D_{44} + d_2^2 D_{66}$	$b_2 c_2 (D_{12} + D_{44})$	$b_2 d_2 (D_{13} + D_{66})$
$b_2 c_2 (D_{21} + D_{44})$	$c_2^2 D_{22} + b_2^2 D_{44} + d_2^2 D_{55}$	$c_2 d_2 (D_{23} + D_{55})$
$b_2 d_2 (D_{31} + D_{66})$	$c_2 d_2 (D_{32} + D_{55})$	$d_2^2 D_{33} + c_2^2 D_{55} + b_2^2 D_{66}$
$b_2 b_3 D_{11} + c_2 c_3 D_{44} + d_2 d_3 D_{66}$	$b_3 c_2 D_{12} + b_2 c_3 D_{44}$	$b_3 d_2 D_{13} + b_2 d_3 D_{66}$
$b_2 c_3 D_{21} + b_3 c_2 D_{44}$	$c_2 c_3 D_{22} + b_2 b_3 D_{44} + d_2 d_3 D_{55}$	$c_3 d_2 D_{23} + c_2 d_3 D_{55}$
$b_2 d_3 D_{31} + b_3 d_2 D_{66}$	$c_2 d_3 D_{32} + c_3 d_2 D_{55}$	$d_2 d_3 D_{33} + c_2 c_3 D_{55} + b_2 b_3 D_{66}$
$b_2 b_4 D_{11} + c_2 c_4 D_{44} + d_2 d_4 D_{66}$	$b_4 c_2 D_{12} + b_2 c_4 D_{44}$	$b_4 d_2 D_{13} + b_2 d_4 D_{66}$
$b_2 c_4 D_{21} + b_4 c_2 D_{44}$	$c_2 c_4 D_{22} + b_2 b_4 D_{44} + d_2 d_4 D_{55}$	$c_4 d_2 D_{23} + c_2 d_4 D_{55}$
$b_2 d_4 D_{31} + b_4 d_2 D_{66}$	$c_2 d_4 D_{32} + c_4 d_2 D_{55}$	$d_2 d_4 D_{33} + c_2 c_4 D_{55} + b_2 b_4 D_{66}$

kolone 10 do 12

$b_1 b_4 D_{11} + c_1 c_4 D_{44} + d_1 d_4 D_{66}$	$b_4 c_1 D_{12} + b_1 c_4 D_{44}$	$b_1 d_1 D_{13} + b_4 d_4 D_{66}$
$b_4 c_1 D_{21} + b_1 c_4 D_{44}$	$c_1 c_4 D_{22} + b_1 b_4 D_{44} + d_1 d_4 D_{55}$	$c_1 d_4 D_{23} + c_4 d_1 D_{55}$
$b_4 d_1 D_{31} + b_1 d_4 D_{66}$	$c_4 d_1 D_{32} + c_1 d_4 D_{55}$	$d_1 d_4 D_{33} + c_1 c_4 D_{55} + b_1 b_4 D_{66}$
$b_2 b_4 D_{11} + c_2 c_4 D_{44} + d_2 d_4 D_{66}$	$b_4 c_2 D_{12} + b_2 c_4 D_{44}$	$b_2 d_2 D_{13} + b_4 d_4 D_{66}$
$b_4 c_2 D_{21} + b_2 c_4 D_{44}$	$c_2 c_4 D_{22} + b_2 b_4 D_{44} + d_2 d_4 D_{55}$	$c_4 d_2 D_{23} + c_2 d_4 D_{55}$
$b_4 d_2 D_{31} + b_2 d_4 D_{66}$	$c_4 d_2 D_{32} + c_2 d_4 D_{55}$	$d_2 d_4 D_{33} + c_2 c_4 D_{55} + b_2 b_4 D_{66}$
$b_3 b_4 D_{11} + c_3 c_4 D_{44} + d_3 d_4 D_{66}$	$b_4 c_3 D_{12} + b_3 c_4 D_{44}$	$b_3 d_3 D_{13} + b_4 d_4 D_{66}$
$b_4 c_3 D_{21} + b_3 c_4 D_{44}$	$c_3 c_4 D_{22} + b_3 b_4 D_{44} + d_3 d_4 D_{55}$	$c_4 d_3 D_{23} + c_3 d_4 D_{55}$
$b_4 d_3 D_{31} + b_3 d_4 D_{66}$	$c_4 d_3 D_{32} + c_3 d_4 D_{55}$	$d_3 d_4 D_{33} + c_3 c_4 D_{55} + b_3 b_4 D_{66}$
$b_4^2 D_{11} + c_4^2 D_{44} + d_4^2 D_{66}$	$b_4 c_4 (D_{12} + D_{44})$	$b_4 d_4 (D_{13} + D_{66})$
$b_4 c_4 (D_{21} + D_{44})$	$c_4^2 D_{22} + b_4^2 D_{44} + d_4^2 D_{55}$	$c_4 d_4 (D_{23} + D_{55})$
$b_4 d_4 (D_{31} + D_{66})$	$c_4 d_4 (D_{32} + D_{55})$	$d_4^2 D_{33} + c_4^2 D_{55} + b_4^2 D_{66}$

KE oblika tetraedra. Linearna interpolacija. CSTh KE

- Matrica raspodele napona KE

$$\mathbf{S} = \mathbf{D}\mathbf{B} = [\mathbf{S}_1 \quad \mathbf{S}_2 \quad \mathbf{S}_3 \quad \mathbf{S}_4]$$

$$\mathbf{S}_i = \frac{E}{6V(1+\nu)(1-2\nu)} \begin{bmatrix} b_i - b_i\nu & c_i\nu & d_i\nu \\ b_i\nu & c_i - c_i\nu & d_i\nu \\ b_i\nu & c_i\nu & d_i - d_i\nu \\ \frac{1}{2}c_i(1-2\nu) & \frac{1}{2}b_i(1-2\nu) & 0 \\ 0 & \frac{1}{2}d_i(1-2\nu) & \frac{1}{2}c_i(1-2\nu) \\ \frac{1}{2}d_i(1-2\nu) & 0 & \frac{1}{2}b_i(1-2\nu) \end{bmatrix}, \quad i=1,2,3,4$$

- Zaključuje se da su pored deformacije i komponente napona u KE konstantne. Ovo ima za posledicu potrebu za velikim brojem KE na mestima naglih promena deformacije i napona što je mana ovog KE
- Vektor ekvivalentnog opterećenja
 - Ukoliko na KE deluju konstantne zapreminske sile

prvi član vektora ekvivalentnog opterećenja

$$\mathbf{q} = \begin{cases} q_{x0} = \text{const.} \\ q_{y0} = \text{const.} \\ q_{z0} = \text{const.} \end{cases}$$

$$\int_V N_1 q_{x0} dV = q_{x0} \int_V N_1^1 N_2^0 N_3^0 N_4^0 dV = q_{x0} \frac{1!0!0!0!}{(1+0+0+0+3)!} 6V = q_{x0} \frac{V}{4}$$

$$\mathbf{Q}^T = \frac{V}{4} \{ q_{x0} \quad q_{y0} \quad q_{z0} \quad q_{x0} \quad q_{y0} \quad q_{z0} \quad q_{x0} \quad q_{y0} \quad q_{z0} \quad q_{x0} \quad q_{y0} \quad q_{z0} \}$$

ukupna zapreminska sila
ravnomerno se deli na svaki čvor KE

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- KE oblika tetraedra sa čvorovima u temenima i sredinama ivica

$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8 \quad \mathbf{d}_9 \quad \mathbf{d}_{10}\}$$

$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \quad i = 1, 2, 3, \dots, 10$$

- Raspodela pomeranja u polju KE definisana je potpunim polinomima drugog stepena

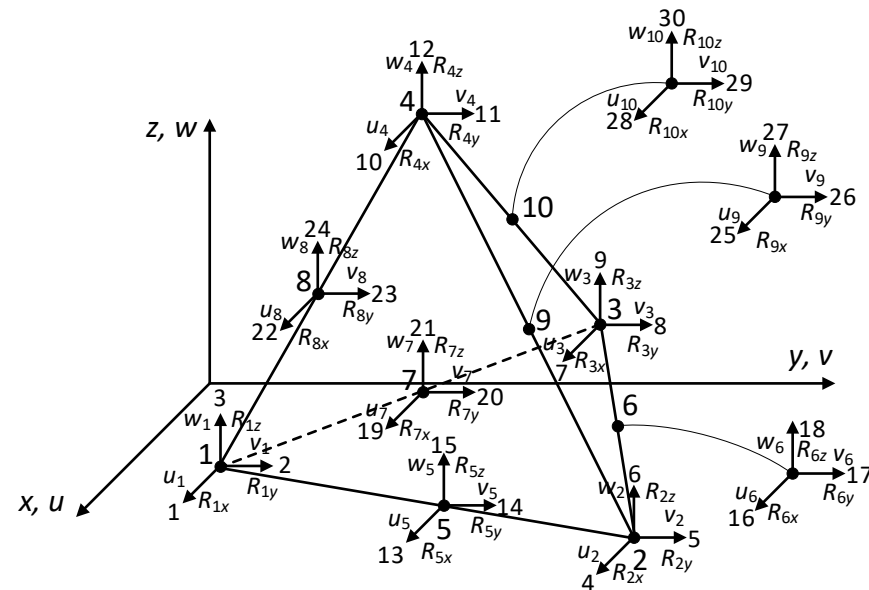
$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz$$

$$v = \alpha_{11} + \alpha_{12} x + \alpha_{13} y + \alpha_{14} z + \alpha_{15} x^2 + \alpha_{16} y^2 + \alpha_{17} z^2 + \alpha_{18} xy + \alpha_{19} yz + \alpha_{20} xz$$

$$w = \alpha_{21} + \alpha_{22} x + \alpha_{23} y + \alpha_{24} z + \alpha_{25} x^2 + \alpha_{26} y^2 + \alpha_{27} z^2 + \alpha_{28} xy + \alpha_{29} yz + \alpha_{30} xz$$

- Matrica IF

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 & N_9 & 0 & 0 & N_{10} \end{bmatrix}$$



KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- IF. Prirodne zapreminske koordinate

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z), \quad i = 1, 2, 3, 4$$

$$N_1 = L_1(2L_1 - 1), \quad N_2 = L_2(2L_2 - 1), \quad N_3 = L_3(2L_3 - 1), \quad N_4 = L_4(2L_4 - 1)$$

$$N_5 = 4L_1 4L_2, \quad N_6 = 4L_2 4L_3, \quad N_7 = 4L_3 4L_4, \quad N_8 = 4L_4 4L_1$$

$$N_9 = 4L_2 4L_4, \quad N_{10} = 4L_3 4L_4$$

- Matrica **B**

$$\mathbf{B} = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \mathbf{B}_3 \quad \cdots \quad \mathbf{B}_{10}] \quad \mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}, \quad i = 1, 2, 3, \dots, 10$$

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

■ Matrica **B**

- S obzirom na to da je $N_i = N_i(L_1(x, y, z), L_2(x, y, z), L_3(x, y, z), L_4(x, y, z))$ koristeći lančano pravilo parcijalnog diferenciranja dobija se

$$\left. \begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial L_1} \frac{\partial L_1}{\partial x} + \frac{\partial N_i}{\partial L_2} \frac{\partial L_2}{\partial x} + \frac{\partial N_i}{\partial L_3} \frac{\partial L_3}{\partial x} + \frac{\partial N_i}{\partial L_4} \frac{\partial L_4}{\partial x} \\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial L_1} \frac{\partial L_1}{\partial y} + \frac{\partial N_i}{\partial L_2} \frac{\partial L_2}{\partial y} + \frac{\partial N_i}{\partial L_3} \frac{\partial L_3}{\partial y} + \frac{\partial N_i}{\partial L_4} \frac{\partial L_4}{\partial y} \\ \frac{\partial N_i}{\partial z} &= \frac{\partial N_i}{\partial L_1} \frac{\partial L_1}{\partial z} + \frac{\partial N_i}{\partial L_2} \frac{\partial L_2}{\partial z} + \frac{\partial N_i}{\partial L_3} \frac{\partial L_3}{\partial z} + \frac{\partial N_i}{\partial L_4} \frac{\partial L_4}{\partial z} \end{aligned} \right\}, \quad i = 1, 2, 3, \dots, 10$$

$$\left. \begin{aligned} \frac{\partial L_i}{\partial x} &= \frac{b_i}{6V} \\ \frac{\partial L_i}{\partial y} &= \frac{c_i}{6V} \\ \frac{\partial L_i}{\partial z} &= \frac{d_i}{6V} \end{aligned} \right\}, \quad i = 1, 2, 3, 4$$

$$\left. \begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{1}{6V} \left(\frac{\partial N_i}{\partial L_1} b_1 + \frac{\partial N_i}{\partial L_2} b_2 + \frac{\partial N_i}{\partial L_3} b_3 + \frac{\partial N_i}{\partial L_4} b_4 \right) \\ \frac{\partial N_i}{\partial y} &= \frac{1}{6V} \left(\frac{\partial N_i}{\partial L_1} c_1 + \frac{\partial N_i}{\partial L_2} c_2 + \frac{\partial N_i}{\partial L_3} c_3 + \frac{\partial N_i}{\partial L_4} c_4 \right) \\ \frac{\partial N_i}{\partial z} &= \frac{1}{6V} \left(\frac{\partial N_i}{\partial L_1} d_1 + \frac{\partial N_i}{\partial L_2} d_2 + \frac{\partial N_i}{\partial L_3} d_3 + \frac{\partial N_i}{\partial L_4} d_4 \right) \end{aligned} \right\}, \quad i = 1, 2, 3, \dots, 10$$

KE oblika tetraedra. Kvadratna interpolacija. LSTh KE

- Nakon određivanja matrice **B** pomoću izraza koristeći matricu elastičnosti i vodeći računa o pravilu za integraciju određuje se matrica krutosti **k**
- U diferencijalnom operatoru pojavljuju se parcijalni izvodi prvog reda pa se zaključuje da je stepen funkcije koja opisuje raspodelu deformacije u polju KE za jedan manji od stepena funkcije koja opisuje raspodelu pomeranja, tj. aproksimacija raspodele deformacije u polju KE je linearna pa se zbog toga element naziva **tetraedar s linearnim deformacijama ili LSTh element** (Linear Strain Tetrahedral element). Razmatrani element **spada u grupu konformnih elemenata**, a **ispunjava i ostale uslove za monotonu konvergenciju rešenja**
- Analognim postupcima opisanim za CSTh element određuje se vektor ekvivalentnog opterećenja, raspodela deformacije i napona za LSTh element

Prizmatični konačni elementi.

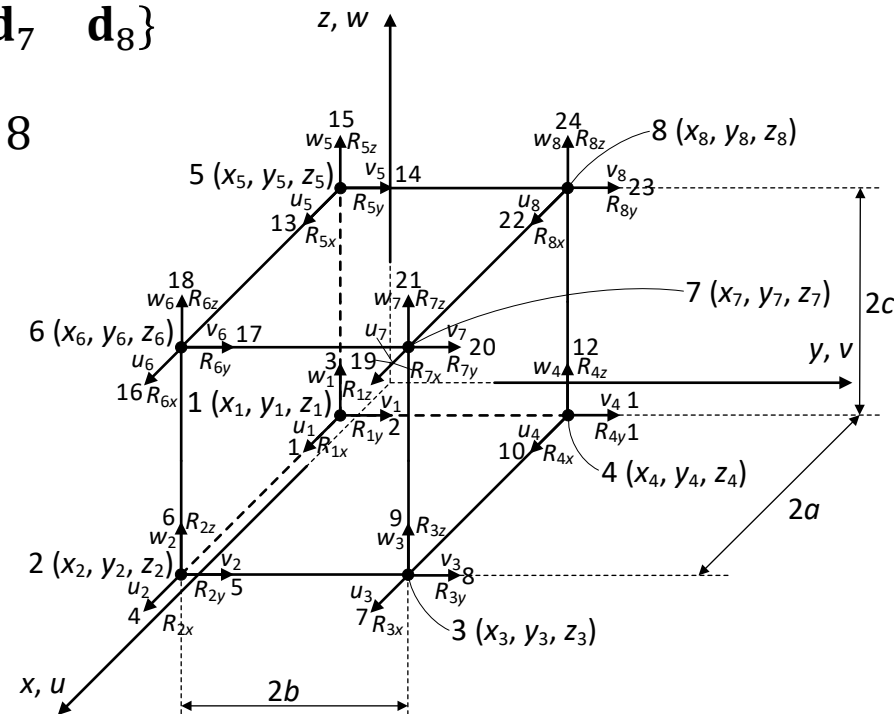
Pravougaoni paralelopiped

- KE oblika pravougaonog paralelopipeda koji ima 8 čvorova

$$\mathbf{d}^T = \{\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3 \quad \mathbf{d}_4 \quad \mathbf{d}_5 \quad \mathbf{d}_6 \quad \mathbf{d}_7 \quad \mathbf{d}_8\}$$

$$\mathbf{d}_i^T = \{u_i \quad v_i \quad w_i\}, \quad i = 1, 2, 3, \dots, 8$$

- Raspodela pomeranja u polju KE definisana je nepotpunim polinomima trećeg stepena, tzv. trilinearna interpolacija pomeranja jer IF sadrže proizvode tri linearna polinoma



$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 yz + \alpha_7 zx + \alpha_8 xyz$$

$$v = \alpha_9 + \alpha_{10} x + \alpha_{11} y + \alpha_{12} z + \alpha_{13} xy + \alpha_{14} yz + \alpha_{15} zx + \alpha_{16} xyz$$

$$w = \alpha_{17} + \alpha_{18} x + \alpha_{19} y + \alpha_{20} z + \alpha_{21} xy + \alpha_{22} yz + \alpha_{23} zx + \alpha_{24} xyz$$

Prizmatični konačni elementi.

Pravougaoni paralelopiped

- IF (analogno kao i kod pravougaonog KE)
 - Lagranžovi polinomi

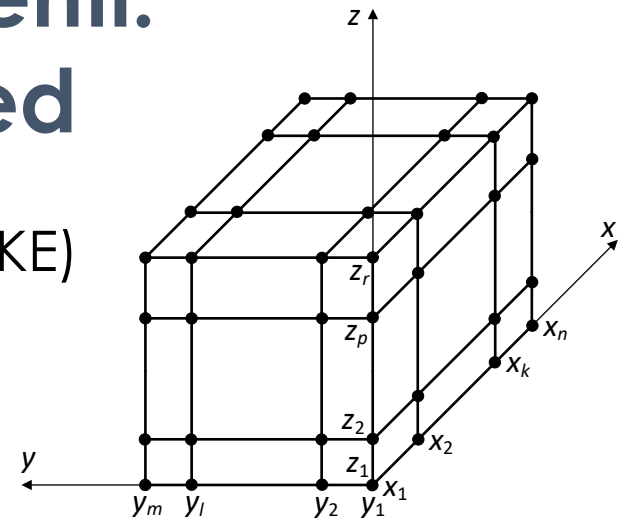
$$N_i(x, y) = L_k(x)L_l(y)L_p(z)$$

$$L_k(x) = \prod_{\substack{j=1 \\ j \neq k}}^n \frac{x - x_j}{x_k - x_j}, \quad L_l(y) = \prod_{\substack{j=1 \\ j \neq l}}^m \frac{y - y_j}{y_l - y_j}, \quad L_p(z) = \prod_{\substack{j=1 \\ j \neq p}}^r \frac{z - z_j}{z_p - z_j}$$

- gde su n, m i r brojevi čvorova u pravcu osa x, y i z , respektivno i $1 \leq k \leq n$, $1 \leq l \leq m$ i $1 \leq p \leq r$
- Vodeći računa o koordinatama početnih i krajnjih čvorova duž ivica KE ($x_1 = -a, x_2 = a, y_1 = -b, y_2 = b, z_1 = -c, z_2 = c$), Lagranžovi polinomi u pravcima osa x, y i z su (slično kao i kod dvodimenzionalnog pravougaonog KE) glase

$$L_1(x) = \frac{1}{2} \left(1 - \frac{x}{a} \right), \quad L_1(y) = \frac{1}{2} \left(1 - \frac{y}{b} \right), \quad L_1(z) = \frac{1}{2} \left(1 - \frac{z}{c} \right)$$

$$L_2(x) = \frac{1}{2} \left(1 + \frac{x}{a} \right), \quad L_2(y) = \frac{1}{2} \left(1 + \frac{y}{b} \right), \quad L_2(z) = \frac{1}{2} \left(1 + \frac{z}{c} \right)$$



Prizmatični konačni elementi. Pravougaoni paralelopiped

■ IF

$$N_1 = L_1(x)L_1(y)L_1(z) = \frac{1}{8}\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)\left(1 - \frac{z}{c}\right)$$

$$N_2 = L_2(x)L_1(y)L_1(z) = \frac{1}{8}\left(1 + \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)\left(1 - \frac{z}{c}\right)$$

$$N_3 = L_2(x)L_2(y)L_1(z) = \frac{1}{8}\left(1 + \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)\left(1 - \frac{z}{c}\right)$$

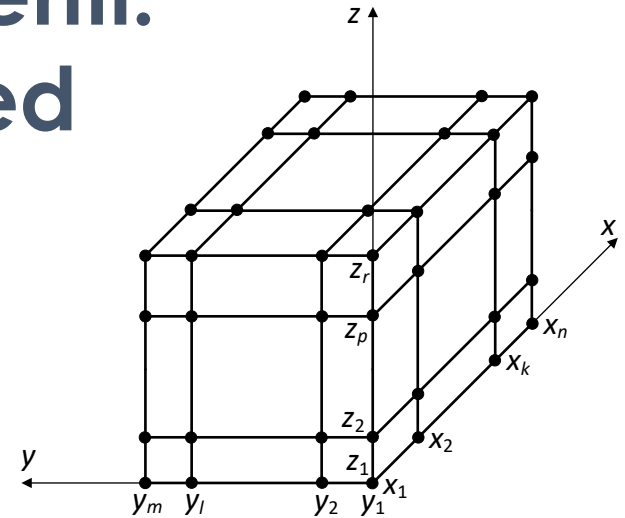
$$N_4 = L_1(x)L_2(y)L_1(z) = \frac{1}{8}\left(1 - \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)\left(1 - \frac{z}{c}\right)$$

$$N_5 = L_1(x)L_1(y)L_2(z) = \frac{1}{8}\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)\left(1 + \frac{z}{c}\right)$$

$$N_6 = L_2(x)L_1(y)L_2(z) = \frac{1}{8}\left(1 + \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)\left(1 + \frac{z}{c}\right)$$

$$N_7 = L_2(x)L_2(y)L_2(z) = \frac{1}{8}\left(1 + \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)\left(1 + \frac{z}{c}\right)$$

$$N_8 = L_1(x)L_2(y)L_2(z) = \frac{1}{8}\left(1 - \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)\left(1 + \frac{z}{c}\right)$$



$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \\ N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 & 0 \\ 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 & 0 \\ 0 & 0 & N_5 & 0 & 0 & N_6 & 0 & 0 & N_7 & 0 & 0 & N_8 \end{bmatrix}$$

■ Matrica krutosti

$$\mathbf{k} = \int_{-c}^c \int_{-b}^b \int_{-a}^a \mathbf{B}^T \mathbf{D} \mathbf{B} dx dy dz$$

Prizmatični konačni elementi. Pravougaoni paralelopiped

■ IF. Prirodne koordinate

$$N_i = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta), \quad i = 1, 2, 3, \dots, 8$$

- gde su $\xi_i = \pm 1$, $\eta_i = \pm 1$ i $\zeta_i = \pm 1$ koordinate čvorova, a veze između prirodnih i lokalnih Dekartovih koordinata glase

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \zeta = \frac{z}{c}$$

■ Sada matrica krutosti glasi

$$\mathbf{k} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta d\zeta$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\det \mathbf{J} = abc$$

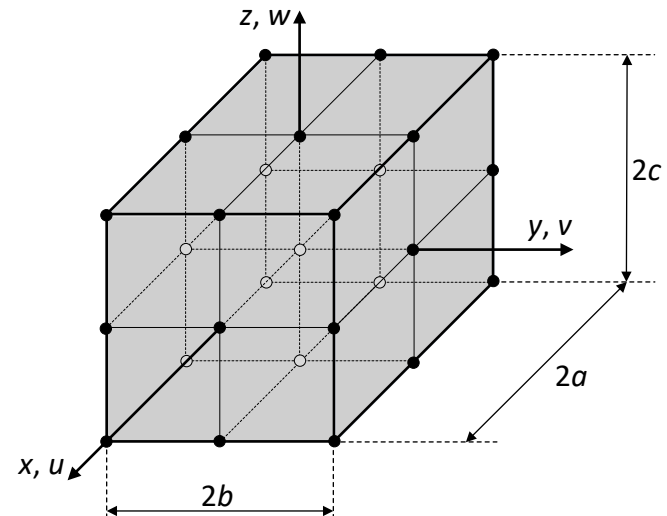
Prizmatični konačni elementi.

Pravougaoni paralelopiped

- Pri određivanju matrice **B** neophodno je odrediti parcijalne izvode IF po lokalnim Dekartovim koordinatama

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix}, \quad i=1,2,3,\dots,8$$

- Lagranžov KE drugog reda (27 čvorova i 81 stepen slobode)**



Prizmatični konačni elementi. Pravougaoni paralelopiped

■ Lagranžov KE drugog reda

- Lagranžovi polinomi za čvorove u pravcu osa x, y i z

$$L_1(x) = \frac{x-x_2}{x_1-x_2} \frac{x-x_3}{x_1-x_3}, \quad L_2(x) = \frac{x-x_1}{x_2-x_1} \frac{x-x_3}{x_2-x_3}, \quad L_3(x) = \frac{x-x_1}{x_3-x_1} \frac{x-x_2}{x_3-x_2}$$

$$L_1(y) = \frac{y-y_2}{y_1-y_2} \frac{y-y_3}{y_1-y_3}, \quad L_2(y) = \frac{y-y_1}{y_2-y_1} \frac{y-y_3}{y_2-y_3}, \quad L_3(y) = \frac{y-y_1}{y_3-y_1} \frac{y-y_2}{y_3-y_2}$$

$$L_1(z) = \frac{z-z_2}{z_1-z_2} \frac{z-z_3}{z_1-z_3}, \quad L_2(z) = \frac{z-z_1}{z_2-z_1} \frac{z-z_3}{z_2-z_3}, \quad L_3(z) = \frac{z-z_1}{z_3-z_1} \frac{z-z_2}{z_3-z_2}$$

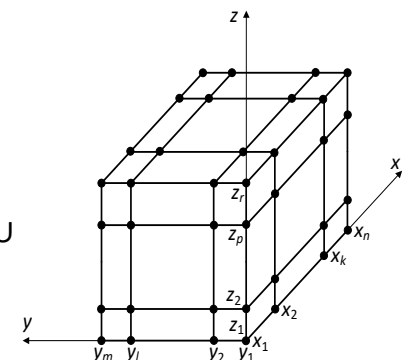
- odnosno vodeći računa o koordinatama čvorova $x_1 = -a$, $x_2 = 0$, $x_3 = a$, $y_1 = -b$, $y_2 = 0$, $y_3 = b$, $z_1 = -c$, $z_2 = 0$ i $z_3 = b$ sledi

$$L_1(x) = \frac{1}{2} \left(\frac{x^2}{a^2} - \frac{x}{a} \right), \quad L_2(x) = 1 - \frac{x^2}{a^2}, \quad L_3(x) = \frac{1}{2} \left(\frac{x^2}{a^2} + \frac{x}{a} \right)$$

$$L_1(y) = \frac{1}{2} \left(\frac{y^2}{b^2} - \frac{y}{b} \right), \quad L_2(y) = 1 - \frac{y^2}{b^2}, \quad L_3(y) = \frac{1}{2} \left(\frac{y^2}{b^2} + \frac{y}{b} \right)$$

$$L_1(z) = \frac{1}{2} \left(\frac{z^2}{c^2} - \frac{z}{c} \right), \quad L_2(z) = 1 - \frac{z^2}{c^2}, \quad L_3(z) = \frac{1}{2} \left(\frac{z^2}{c^2} + \frac{z}{c} \right)$$

Analognim postupkom kao i kod pravougaonog KE koristeći šemu i izraz $N_i(x, y, z) = L_k(x)L_l(y)L_p(z)$ određuju se IF



Prizmatični konačni elementi.

Pravougaoni paralelopiped

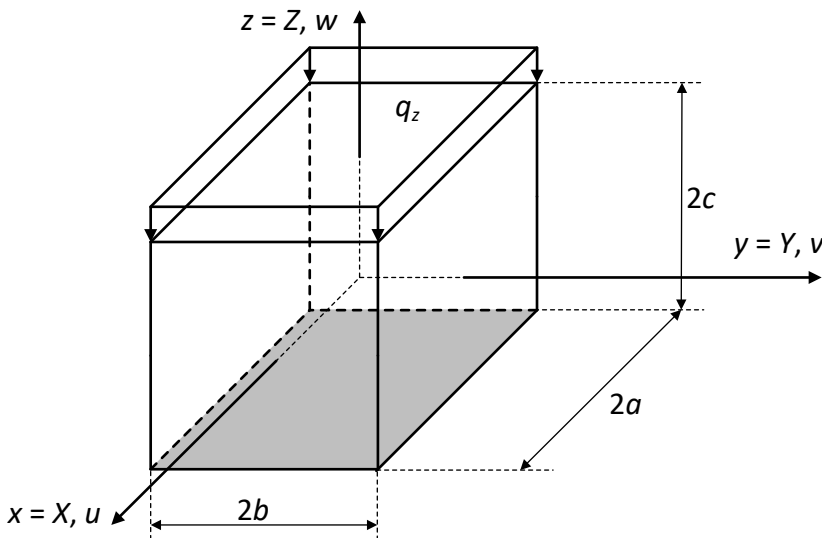
- S obzirom na to da Lagranžovi KE višeg reda imaju unutrašnje čvorove, kod njih se javlja veliki broj stepeni slobode
- Slično kao i kod dvodimenzionalnih KE, nepotpunost polinoma za aproksimaciju pomeranja u polju trodimenzionalnog Lagranžovog KE je nedostatak ali im je dobra osobina jednostavnost izvođenja IF
- Pored elemenata sa jednakim brojem čvorova u pravcu Dekartovih osa mogu da se izvedu elementi s različitim brojem čvorova u pravcu koordinatnih osa množenjem Lagranžovih polinoma različitog stepena

Prizmatični konačni elementi. Pravougaoni paralelopiped

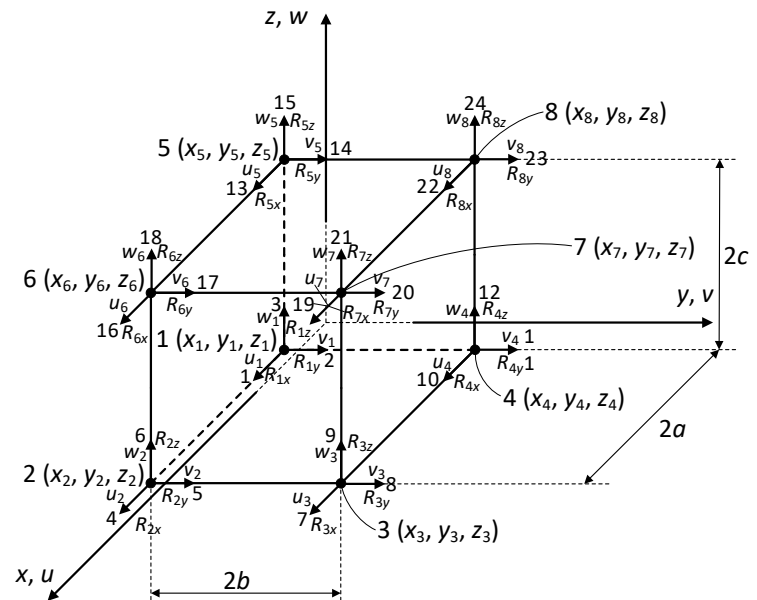
■ Primer

- Podaci su: $q_z = 10 \text{ MPa}$, modul elastičnosti $E = 30 \text{ GPa}$, Poasonov koeficijent $\nu = 0,2$, $a = b = c = 0,5 \text{ m}$

Matematički model 3D tela oslonjenog u svim pravcima po površini jedne stranice (zatomnjena površina). Potrebno je odrediti pomeranje temena opterećene stranice



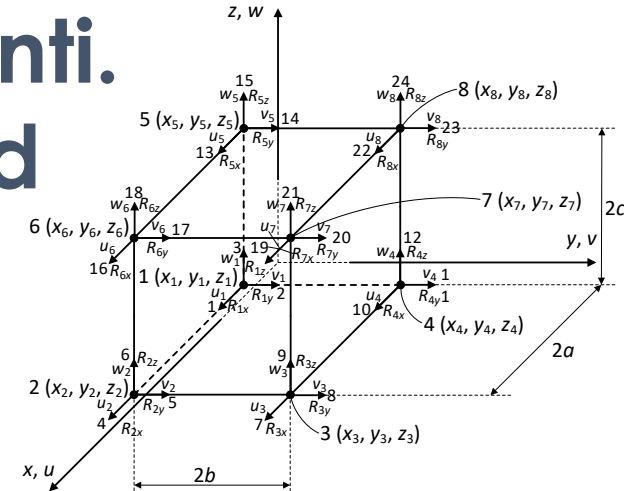
Matematički model je diskretizovan jednim 3D KE koji ima 8 čvorova. Postoji simetrija ali nije korišćena pri analizi



Prizmatični konačni elementi. Pravougaoni paralelopiped

Primer

$$\mathbf{d} = \mathbf{d}^* = \begin{Bmatrix} d_1^* \\ \vdots \\ d_8^* \end{Bmatrix} = \begin{Bmatrix} u_1 & 1 \\ v_1 & 2 \\ w_1 & 3 \\ \vdots & \vdots \\ u_8 & 22 \\ v_8 & 23 \\ w_8 & 24 \end{Bmatrix} \quad \mathbf{d}_a^* = \begin{Bmatrix} u_5 & 13 \\ v_5 & 14 \\ w_5 & 15 \\ \vdots & \vdots \\ u_8 & 22 \\ v_8 & 23 \\ w_8 & 24 \end{Bmatrix}$$



$$\mathbf{K}_{aa}^* \mathbf{d}_a^* = \mathbf{S}_a^* = \mathbf{P}_a^* + \mathbf{Q}_a^* \Rightarrow \mathbf{d}_a^* = \mathbf{K}_{aa}^{*-1} \mathbf{S}_a^*$$

Matrica **B** je prikazana samo za jedna čvor

$$\mathbf{D} = \begin{bmatrix} 3,3 & 0,83 & 0,83 & 0 & 0 & 0 \\ & 3,3 & 0,83 & 0 & 0 & 0 \\ & & 3,3 & 0 & 0 & 0 \\ & & & 1,25 & 0 & 0 \\ & & & & 1,25 & 0 \\ \text{sim.} & & & & & 1,25 \end{bmatrix} \cdot 10^7$$

$$\mathbf{B}_1 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} \end{bmatrix}$$

$$\frac{\partial N_1}{\partial x} = -\frac{(b-y)(c-z)}{8abc} = -0,25(1-2y)(1-2z)$$

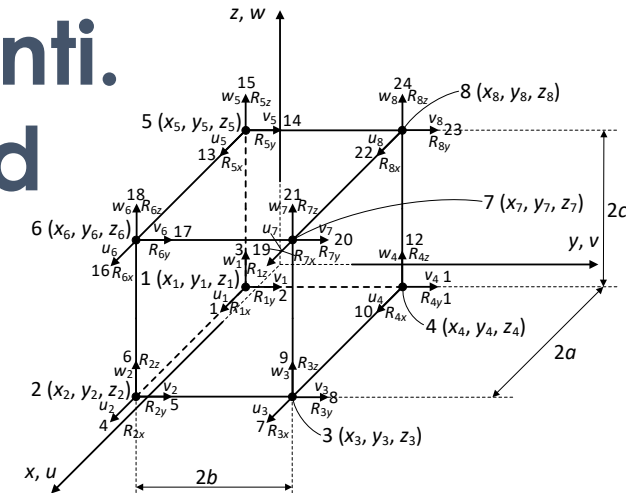
$$\frac{\partial N_1}{\partial y} = -\frac{(a-x)(c-z)}{8abc} = -0,25(1-2x)(1-2z)$$

$$\frac{\partial N_1}{\partial z} = -\frac{(a-x)(b-y)}{8abc} = -0,25(1-2x)(1-2y)$$

Prizmatični konačni elementi. Pravougaoni paralelopiped

Primer

$$\mathbf{k} = \mathbf{k}^* = \begin{bmatrix} 1 & 2 & \dots & 23 & 24 \\ 6,4815 & 1,7361 & \dots & 0,1736 & 0,1736 \\ & 6,4815 & \dots & -2,1991 & -1,7361 \\ & & \ddots & \vdots & \vdots \\ & & & 6,4815 & 1,7361 \\ \text{sim.} & & & & 6,4815 \end{bmatrix}_{24 \times 24} \cdot 10^6$$



IF za opterećenu površinu ($z = c = 0,5 \text{ m}$) glase

$$N_{s1} = N_{s2} = N_{s3} = N_{s4} = 0$$

$$N_{s5} = \frac{0,25(a-x)(b-y)}{ab} = 0,25(1-2x)(1-2y)$$

$$N_{s6} = \frac{0,25(a+x)(b-y)}{ab} = 0,25(1+2x)(1-2y)$$

$$N_{s7} = \frac{0,25(a+x)(b+y)}{ab} = 0,25(1+2x)(1+2y)$$

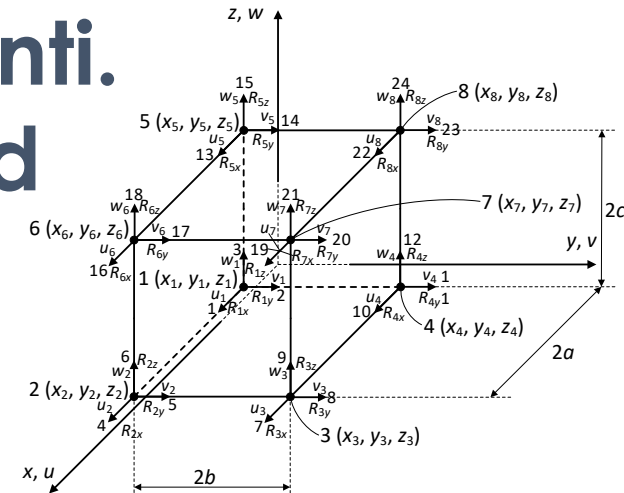
$$N_{s8} = \frac{0,25(a-x)(b+y)}{ab} = 0,25(1-2x)(1+2y)$$

$$\begin{Bmatrix} Q_{5z} \\ Q_{6z} \\ Q_{7z} \\ Q_{8z} \end{Bmatrix} = \begin{Bmatrix} Q_{5z}^* \\ Q_{6z}^* \\ Q_{7z}^* \\ Q_{8z}^* \end{Bmatrix} = \int_{-b-a}^{-b-a} \int_{-b-a}^{-b-a} \begin{Bmatrix} N_{s5}(x,y) \\ N_{s6}(x,y) \\ N_{s7}(x,y) \\ N_{s8}(x,y) \end{Bmatrix} q_z(x,y) dx dy = - \begin{Bmatrix} 2500,0 & 15 \\ 2500,0 & 18 \\ 2500,0 & 21 \\ 2500,0 & 24 \end{Bmatrix}$$

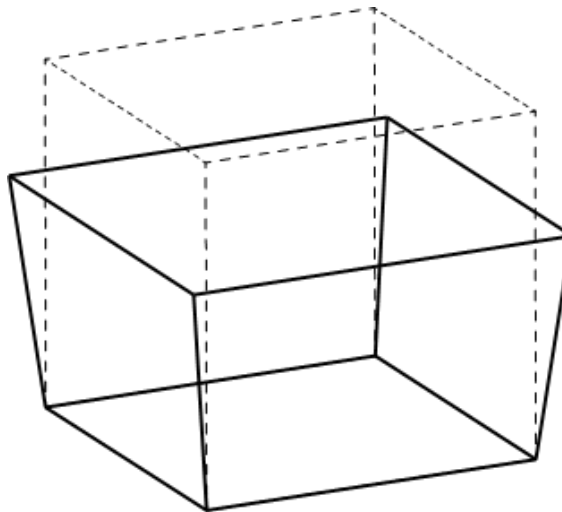
Prizmatični konačni elementi. Pravougaoni paralelopiped

Primer

$$K_{aa}^* = \begin{bmatrix} 13 & 14 & \dots & 23 & 24 \\ 6,4815 & 1,7361 & \dots & 0,3472 & -0,8681 \\ & 6,4815 & \dots & -2,3148 & 0,3472 \\ & & \ddots & \vdots & \vdots \\ & & & 6,4815 & 1,7361 \\ \text{sim.} & & & & 6,4815 \end{bmatrix} \cdot 10^6 \quad \mathbf{Q}_a^* = \mathbf{S}_a^* = - \begin{bmatrix} 0 & 13 \\ 0 & 14 \\ 2500,0 & 15 \\ \vdots & \vdots \\ 0 & 22 \\ 0 & 23 \\ 2500,0 & 24 \end{bmatrix} \cdot 10^3$$



$$\mathbf{d}_a^* = \mathbf{K}_{aa}^{*-1} \mathbf{S}_a^* = \begin{bmatrix} -0,0450 & 13 \\ -0,0450 & 14 \\ -0,3225 & 15 \\ 0,0450 & 16 \\ -0,0450 & 17 \\ -0,3225 & 18 \\ 0,0450 & 19 \\ 0,0450 & 20 \\ -0,3225 & 21 \\ -0,0450 & 22 \\ 0,0450 & 23 \\ -0,3225 & 24 \end{bmatrix} \cdot 10^{-3} \text{ m}$$



Komentari:

- Konvergencija ka tačnom rešenju može da se postigne povećanjem broja KE u modelu (progušćenje mreže) i/ili primenom KE višeg reda
- Pomeranje temena opterećene stranice u pravcu z ose, određeno primenom KE oblika pravougaonog paralelopipeda koji imaju 8, 20, 21 i 27 čvorova i KE oblika tetraedra koji imaju 4, 10 i 11 čvorova u modelima sa uniformnom i adaptivnom mrežom, konvergira ka vrednosti približno jednakoj $-0,331 \cdot 10^{-3} \text{ m}$

Prizmatični konačni elementi.

Pravougaoni paralelopiped

■ Primer

- Primena prirodnih koordinata pojednostavljuje određivanje matrice krutosti KE i vektora ekvivalentnog opterećenja

$$N_i = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta), \quad i = 1, 2, 3, \dots, 8$$

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix}, \quad i = 1, 2, 3, \dots, 8$$

$$\mathbf{k} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta d\zeta \quad \det \mathbf{J} = abc$$

Vodeći računa da je za opterećenu površinu prirodna koordinata $\zeta = 1$

$$\begin{Bmatrix} Q_{5z} \\ Q_{6z} \\ Q_{7z} \\ Q_{8z} \end{Bmatrix} = \int_{-1}^1 \int_{-1}^1 \begin{Bmatrix} N_{s5}(\xi, \eta) \\ N_{s6}(\xi, \eta) \\ N_{s7}(\xi, \eta) \\ N_{s8}(\xi, \eta) \end{Bmatrix} q_z(\xi, \eta) \det \mathbf{J} d\xi d\eta \quad \det \mathbf{J} = ab$$

Prizmatični konačni elementi.

Serendipiti KE

- Slično kao i kod dvodimenzionalnih KE, imaju čvorove samo duž ivica i na taj način smanjuje se broj stepeni slobode u odnosu na Lagranžove KE pri istoj raspodeli pomeranja
- Prednost u odnosu na Lagranžove KE je i u manjoj nepotpunosti polinoma ali imaju složeniji postupak izvođenja IF
- U Serendipiti familiju spada i KE u obliku pravougaonog paralelopipeda, tzv. linearni element (8 čvorova i 24 stepena slobode)

Prizmatični konačni elementi.

Serendipiti KE

- Kvadratni (paraboličan) element (20 čvorova i 60 stepeni slobode)

- IF

- za čvorove u temenima

$$N_i = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta)(\xi_i \xi + \eta_i \eta + \zeta_i \zeta - 2)$$

- gde su $\xi_i = \pm 1$, $\eta_i = \pm 1$ i $\zeta_i = \pm 1$ koordinate čvorova
 - i za čvorove u sredinama ivica

$$\xi_i = 0, \eta_i = \pm 1 \text{ i } \zeta_i = \pm 1:$$

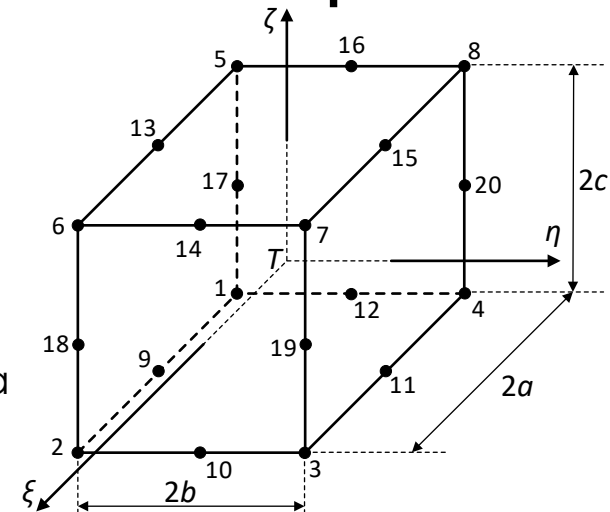
$$N_i = \frac{1}{4}(1 - \xi^2)(1 + \eta_i \eta)(1 + \zeta_i \zeta)$$

$$\xi_i = \pm 1, \eta_i = 0 \text{ i } \zeta_i = \pm 1:$$

$$N_i = \frac{1}{4}(1 + \xi_i \xi)(1 - \eta^2)(1 + \zeta_i \zeta)$$

$$\text{ i } \xi_i = \pm 1, \eta_i = \pm 1 \text{ i } \zeta_i = 0:$$

$$N_i = \frac{1}{4}(1 + \xi_i \xi)(1 + \eta^2)(1 - \zeta^2)$$



Izoparametarski konačni elementi. KE oblika heksaedra

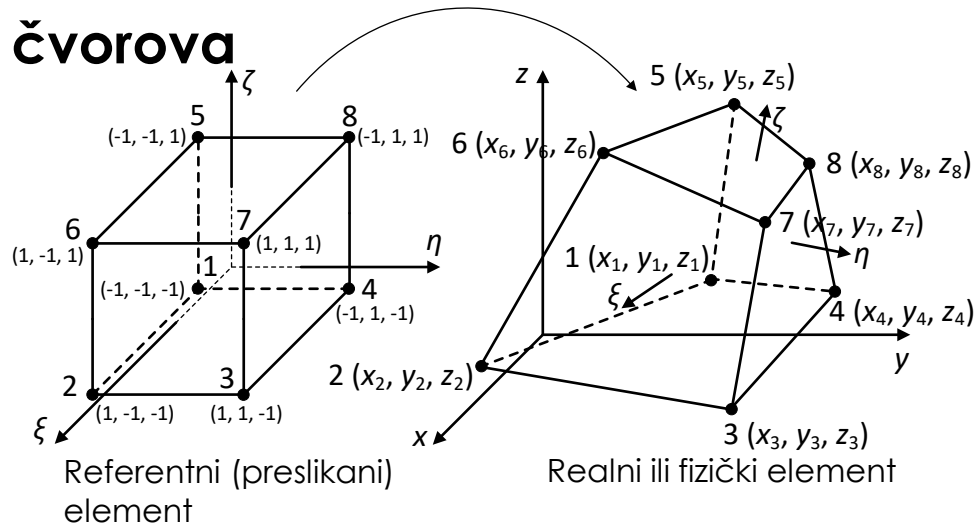
- Heksaedarski KE koji ima 8 čvorova
- Geometrija i raspodela pomeranja proizvoljnog heksaedarskog KE definišu se na sledeći način

$$x = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) x_i, \quad y = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) y_i, \quad z = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) z_i$$

$$u = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) u_i, \quad v = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) v_i, \quad w = \sum_{i=1}^8 N_i(\xi, \eta, \zeta) w_i$$

- gde su x_i , y_i i z_i Dekartove koordinate i -tog čvora realnog elementa, a u_i , v_i i w_i komponente pomeranja i -tog čvora realnog elementa u pravcu Dekartovih osa
- Funkcije preslikavanja i IF su

$$N_i = \frac{1}{8}(1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta), \quad i = 1, 2, 3, \dots, 8$$



Izoparametarski konačni elementi. KE oblika heksoedra

- Parcijalni izvodi IF po Dekartovim koordinatama određuju se prema izrazu

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} \quad \mathbf{J} = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \eta} y_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \eta} z_i \\ \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \zeta} x_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \zeta} y_i & \sum_{i=1}^{N_{\text{vor}}} \frac{\partial N_i}{\partial \zeta} z_i \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

$$\det \mathbf{J} = -J_{13}J_{22}J_{31} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{11}J_{23}J_{32} - J_{12}J_{21}J_{33} + J_{11}J_{22}J_{33}$$

$$dxdydz = \det \mathbf{J} d\xi d\eta d\zeta$$

- **Matrica krutosti**

$$\mathbf{k} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta d\zeta$$

Zbog složenosti podintegralnih funkcija elementi matrice krutosti određuju se najčešće numeričkom integracijom

Izoparametarski konačni elementi. KE oblika heksaedra

Matrica B

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_{N_{\text{čvor}}} \end{bmatrix}$$

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}, \quad i = 1, 2, 3, \dots, N_{\text{čvor}}$$

IF za heksaedarski KE koji ima 8 do 20 čvorova

$$N_1 = g_1 - \frac{g_9 + g_{12} + g_{17}}{2}, \quad N_2 = g_2 - \frac{g_9 + g_{10} + g_{18}}{2}, \quad N_3 = g_3 - \frac{g_{10} + g_{11} + g_{19}}{2}$$

$$N_4 = g_4 - \frac{g_{11} + g_{12} + g_{20}}{2}, \quad N_5 = g_5 - \frac{g_{13} + g_{16} + g_{17}}{2}, \quad N_6 = g_6 - \frac{g_{13} + g_{14} + g_{18}}{2}$$

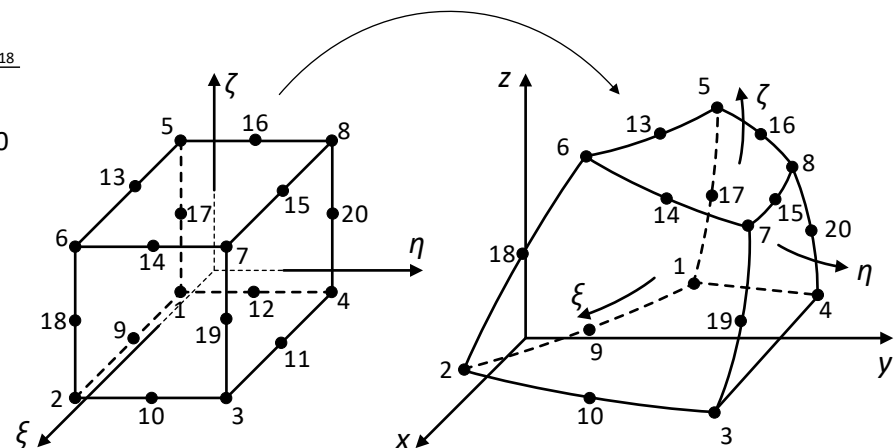
$$N_7 = g_7 - \frac{g_{14} + g_{15} + g_{19}}{2}, \quad N_8 = g_8 - \frac{g_{15} + g_{16} + g_{20}}{2}, \quad N_j = g_j, \quad j = 9, 10, \dots, 20$$

$$g_i = N(\xi, \xi_i) N(\eta, \eta_i) N(\zeta, \zeta_i)$$

$$\xi = \pm 1 \rightarrow N = \frac{1}{2}(1 + \xi \xi_i), \quad \xi = 0 \rightarrow N = \frac{1}{2}(1 - \xi^2)$$

$$\eta = \pm 1 \rightarrow N = \frac{1}{2}(1 + \eta \eta_i), \quad \eta = 0 \rightarrow N = \frac{1}{2}(1 - \eta^2)$$

$$\zeta = \pm 1 \rightarrow N = \frac{1}{2}(1 + \zeta \zeta_i), \quad \zeta = 0 \rightarrow N = \frac{1}{2}(1 - \zeta^2)$$

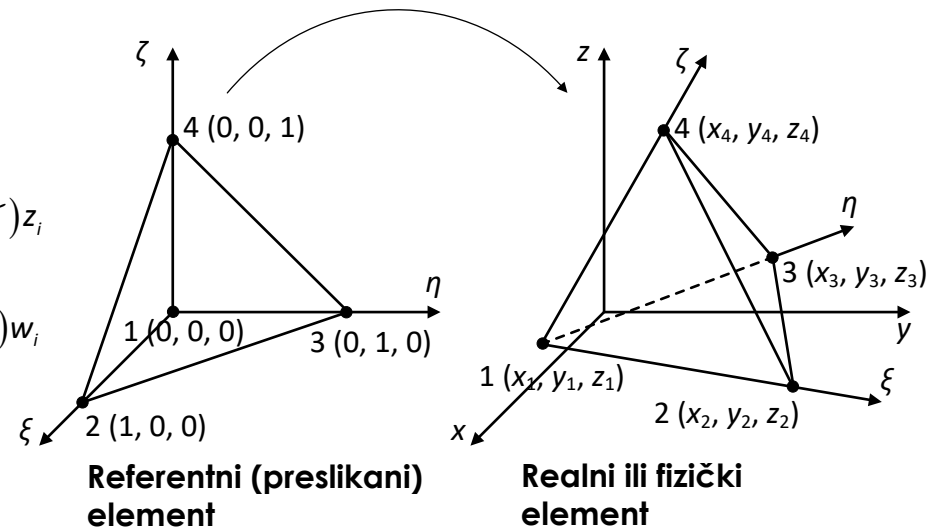


Izoparametarski konačni elementi. KE oblika tetraedra sa 4 čvora

- Geometrija i raspodela pomeranja opisani su sledećim izrazima

$$x = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) x_i, \quad y = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) y_i, \quad z = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) z_i$$

$$u = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) u_i, \quad v = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) v_i, \quad w = \sum_{i=1}^4 N_i(\xi, \eta, \zeta) w_i$$



- IF su jednake zapreminskim koordinatama

$$N_1 = L_1 = 1 - \xi - \eta - \zeta, \quad N_2 = L_2 = \xi$$

$$N_3 = L_3 = \eta, \quad N_4 = L_4 = \zeta$$

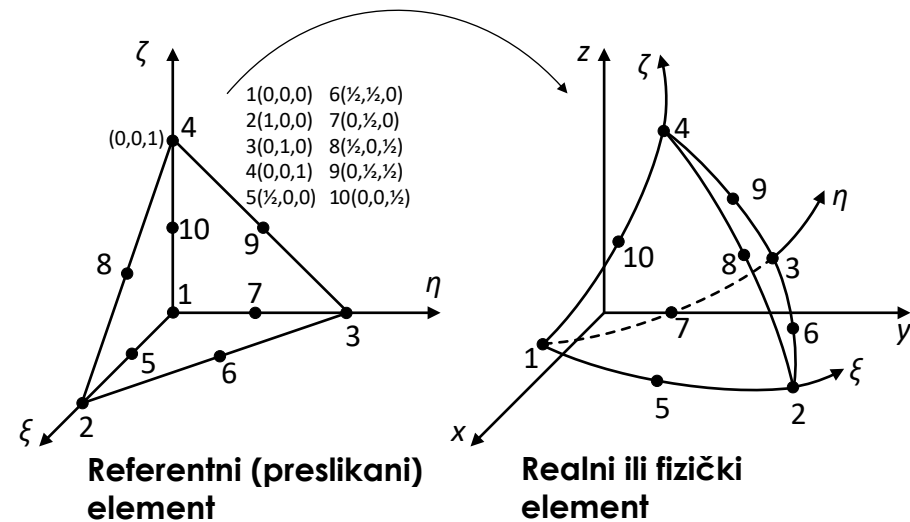
Izoparametarski konačni elementi.

KE oblika tetraedra sa 10 čvorova

- ili KE drugog reda oblika tetraedra
- Geometrija i raspodela pomeranja opisani su sledećim izrazima

$$x = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) x_i, \quad y = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) y_i, \quad z = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) z_i$$

$$u = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) u_i, \quad v = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) v_i, \quad w = \sum_{i=1}^{10} N_i(\xi, \eta, \zeta) w_i$$



- IF

$$N_1 = L_1(2L_1 - 1), \quad N_2 = L_2(2L_2 - 1)$$

$$N_3 = L_3(2L_3 - 1), \quad N_4 = L_4(2L_4 - 1)$$

$$N_5 = 4L_1L_2, \quad N_6 = 4L_2L_3, \quad N_7 = 4L_2L_3$$

$$N_8 = 4L_2L_4, \quad N_9 = 4L_3L_4, \quad N_{10} = 4L_1L_4$$

$$L_1 = 1 - \xi - \eta - \zeta,$$

$$L_2 = \xi$$

$$L_3 = \eta,$$

$$L_4 = \zeta$$

Izoparametarski konačni elementi. KE oblika tetraedra

■ IF za KE od 4 do 10 čvorova

$$N_1 = 1 - \xi - \eta - \zeta - \frac{1}{2}N_5 - \frac{1}{2}N_7 - \frac{1}{2}N_{10}$$

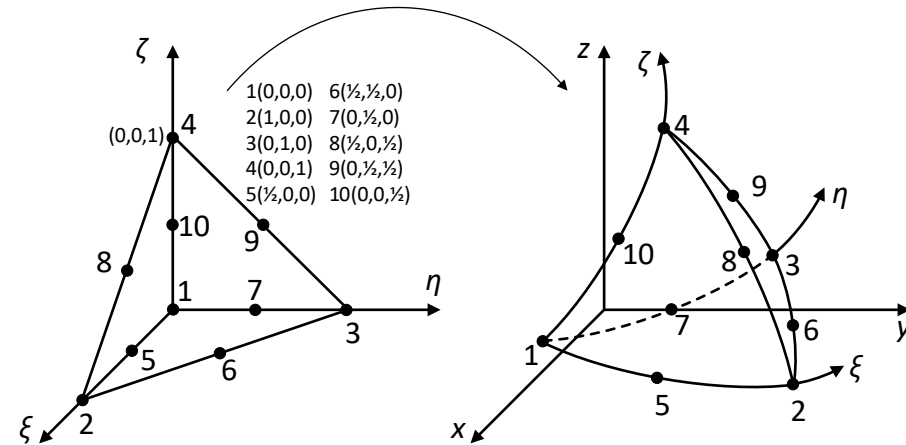
$$N_2 = \xi - \frac{1}{2}N_5 - \frac{1}{2}N_6 - \frac{1}{2}N_8$$

$$N_3 = \eta - \frac{1}{2}N_6 - \frac{1}{2}N_7 - \frac{1}{2}N_9$$

$$N_4 = \zeta - \frac{1}{2}N_8 - \frac{1}{2}N_9 - \frac{1}{2}N_{10}$$

$$N_5 = 4\xi(1 - \xi - \eta - \zeta), \quad N_6 = 4\xi\eta, \quad N_7 = 4\xi(1 - \xi - \eta - \zeta)$$

$$N_8 = 4\xi\zeta, \quad N_9 = 4\eta\zeta, \quad N_{10} = 4\zeta(1 - \xi - \eta - \zeta)$$



■ Parcijalni izvodi

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} \quad \mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \xi} y_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \xi} z_i \\ \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \eta} y_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \eta} z_i \\ \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \zeta} x_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \zeta} y_i & \sum_{i=1}^{N_{\text{čvor}}} \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22}J_{33} - J_{23}J_{32} & J_{13}J_{32} - J_{12}J_{33} & J_{12}J_{23} - J_{13}J_{22} \\ J_{23}J_{31} - J_{21}J_{33} & J_{11}J_{33} - J_{13}J_{31} & J_{13}J_{21} - J_{11}J_{23} \\ J_{21}J_{32} - J_{22}J_{31} & J_{12}J_{31} - J_{11}J_{32} & J_{11}J_{22} - J_{12}J_{21} \end{bmatrix}$$

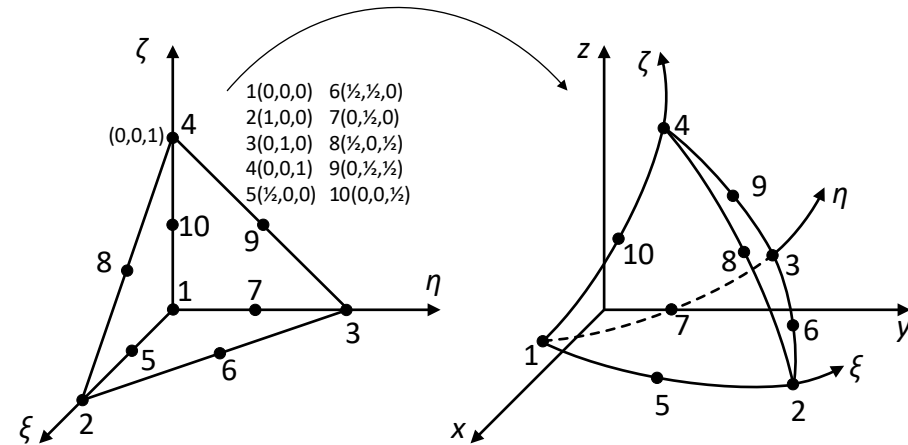
$$\det \mathbf{J} = -J_{13}J_{22}J_{31} + J_{12}J_{23}J_{31} + J_{13}J_{21}J_{32} - J_{11}J_{23}J_{32} - J_{12}J_{21}J_{33} + J_{11}J_{22}J_{33}$$

Izoparametarski konačni elementi. KE oblika tetraedra

Matrica B

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \dots & \mathbf{B}_{N_{\text{čvor}}} \end{bmatrix}$$

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix}, \quad i = 1, 2, 3, \dots, N_{\text{čvor}}$$



Matrica krutosti

$$\mathbf{k} = \int_{\xi=0}^{\xi=1} \int_{\eta=0}^{\eta=1-\xi} \int_{\zeta=0}^{\zeta=1-\xi-\eta} \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\zeta d\eta d\xi$$

Zbog složenosti podintegralnih funkcija elementi matrice krutosti određuju se najčešće **numeričkom integracijom**